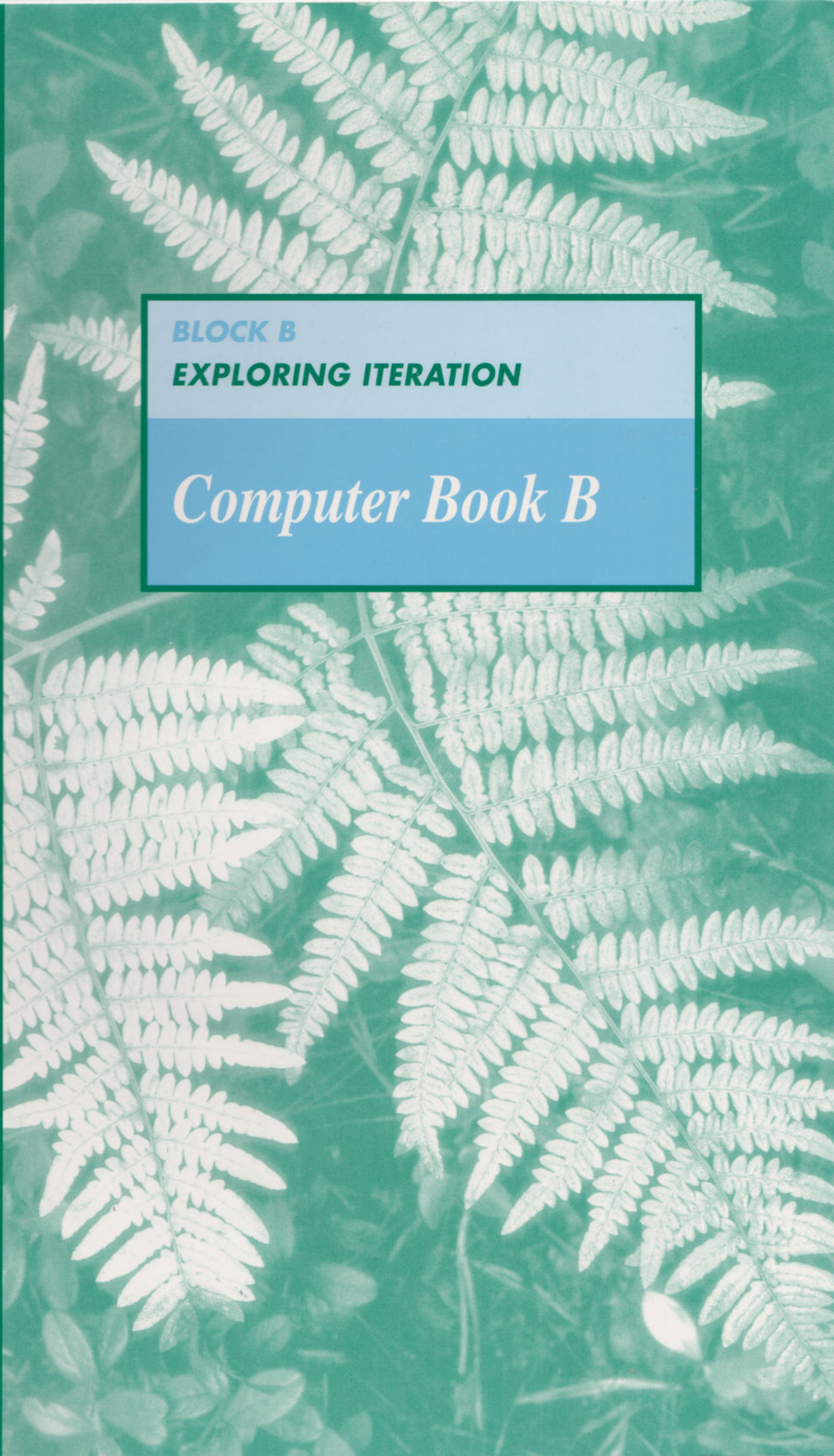
MS221 CB B

The Open University

A second level interdisciplinary course

Exploring Mathematics



BLOCK B
EXPLORING ITERATION

Computer Book B

COMPUTER BOOK

B

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EXPLORING ITERATION

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Prepared by the course team

About this course

This computer book forms part of the course MS221 *Exploring Mathematics*. This course and the courses MU120 *Open Mathematics* and MST121 *Using Mathematics* provide a flexible means of entry to university-level mathematics. Further details may be obtained from the address below.

MS221 uses the software package Mathcad (MathSoft, Inc.) to investigate mathematical concepts and as a tool in problem solving. This software is provided as part of the course.

This publication forms part of an Open University course. Details of this and other Open University courses can be obtained from the Course Information and Advice Centre, PO Box 724, The Open University, Milton Keynes, MK7 6ZS, United Kingdom: tel. +44 (0)1908 653231, e-mail general-enquiries@open.ac.uk

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3.1

Contents

Guidance notes

This computer book contains those sections of the chapters in Block B which require you to use Mathcad. Each of these chapters contains instructions as to when you should first refer to particular material in this computer book, so you are advised not to work on the activities here until you have reached the appropriate points in the chapters.

In order to use this computer book, you will need the following Mathcad files.

Chapter B1

221B1-01 Iterations of $f(x) = x^2 + c$
221B1-02 Overview of iterations of $f(x) = x^2 + c$
221B1-03 Iterating real functions

Chapter B2

221B2-01 Linear transformations
221B2-02 Affine transformations

Chapter B3

221B3-01 Iterating linear transformations
221B3-02 Iterating affine transformations
221B3-03 Iterated function systems (Optional)

Instructions for installing these files onto your computer's hard disk, and for opening them, are given in Chapter A0 of MST121.

The computer activities for each chapter also require you to work with Mathcad documents which you have created yourself.

Activities based on software vary both in nature and in length. Sometimes the instructions for an activity appear only in the computer book; in other cases, instructions are given in the computer book and on screen. Feedback on an activity is sometimes provided on screen and sometimes given in the computer book.

For advice on how each computer session fits into suggested study patterns, refer to the Study guides in the chapters.

Chapter B1, Section 4
Iterating real functions with the computer

In this section you will explore iteration sequences, using Mathcad worksheets that draw graphical iteration diagrams and calculate related information, such as fixed points and gradients.

In Subsection 4.1, you will explore sequences obtained by iterating quadratic functions of the form $f(x) = x^2 + c$. You will see that many different types of behaviour occur; for example, such sequences can tend to p-cycles for various periods p. This subsection is quite long; if you are short of time, then concentrate on Activities 4.1 and 4.2.

The concept of a p-cycle with $p > 2$ was defined in the main text.

In Subsection 4.2, Mathcad is used to provide an overview of how the behaviour of these iteration sequences changes as c varies, and you will see that the changes in behaviour are surprisingly complicated. Subsection 4.3 is *optional*; it contains a discussion of certain other quadratic iteration sequences. In Subsection 4.4, you will use a Mathcad worksheet that allows you to study the sequences obtained by iterating *any* real function f.

4.1 Sequences obtained by iterating $f(x) = x^2 + c$

In this subsection you will explore sequences obtained by iterating functions of the form $f(x) = x^2 + c$, where c is a parameter whose value can be varied.

You have seen that if we wish to study the long-term behaviour of iteration sequences, then it is useful to find and classify any fixed points and 2-cycles of the function.

The fixed point equation and the 2-cycle equation of the function $f(x) = x^2 + c$ can be solved to give formulas, in terms of c, for the fixed points and for those points that are members of a 2-cycle. The Mathcad worksheet that accompanies this subsection uses these formulas to calculate any fixed points and 2-cycles of f. The number of fixed points and 2-cycles depends on the value of c, but f always has at most two fixed points, and at most one 2-cycle.

You saw in the main text that a fixed point a can be classified according to the value of $f'(a)$, the gradient of the graph of f at $(a, f(a))$. The gradient formula for the function $f(x) = x^2 + c$ is $f'(x) = 2x$; in particular, for a fixed point a, we have $f'(a) = 2a$.

Recall that a fixed point a is
attracting if $|f'(a)| < 1$;
repelling if $|f'(a)| > 1$;
indifferent if $|f'(a)| = 1$.

Similarly, a 2-cycle a, b of f can be classified according to the value of the gradient product $f'(a)f'(b)$; this is equal to the gradient of the graph of the composite function $f \circ f$ at a, and also at b. For a 2-cycle a, b of the function $f(x) = x^2 + c$, we have $f'(a)f'(b) = (2a)(2b) = 4ab$.

A 2-cycle a, b is
attracting if $|f'(a)f'(b)| < 1$;
repelling if $|f'(a)f'(b)| > 1$;
indifferent if $|f'(a)f'(b)| = 1$.

The first activity introduces the Mathcad worksheet that you will be using.

Activity 4.1 Graphical iteration on the computer

Remember to create your own working copy of the Mathcad file.

Open Mathcad file **221B1-01 Iterations of f(x)=x^2+c**, and move to page 2 of the worksheet. Here N iterations of the function $f(x) = x^2 + c$ are carried out, with initial term x_0. The resulting sequence x_n is displayed on a graphical iteration diagram, and the last eleven terms calculated are listed in a table. Values for the fixed points, the 2-cycle, and the associated gradients and gradient product are displayed (to three decimal places) underneath the table.

The page is set up with $c = 0$, so the function is initially $f(x) = x^2$. If you look at the fixed point and 2-cycle information, then you will see that Mathcad calculates that this function f has two fixed points, 0 and 1. The values of the gradients show that 0 is attracting whereas 1 is repelling. The 2-cycle variables indicate that f has no 2-cycle.

The variables M and p will be introduced in Activities 4.2 and 4.5, respectively. You should not alter the values of these for now.

The page is also set up with $N = 10$ and $x_0 = 0.8$. In parts (a) to (d) below you are asked to try other values for these two variables.

Remember that to *set* variables in Mathcad we define them using the symbol := , which can be obtained from the appropriate button on the 'Calculator' toolbar or by typing : (a colon, given by `[Shift];`). The symbol = *evaluates* variables in Mathcad.

(a) Describe the long-term behaviour of the iteration sequence with initial term $x_0 = 0.8$. Then set x_0 to each of the values -0.8, 0 and 1.1 in turn, and in each case describe the long-term behaviour of x_n.

(b) Set $x_0 = 0.999$, which is a number just less than the repelling fixed point. Then set $N = 100$, and describe the long-term behaviour of x_n.

(c) With N still set to 100, set $x_0 = 1.001$, which is a number just greater than the repelling fixed point. Try to explain why Mathcad behaves as it does.

(d) With x_0 still set to 1.001, set $N = 15$, and describe the long-term behaviour of x_n.

Comment

(a) When $x_0 = 0.8$, the sequence converges to the fixed point 0, with a staircase pattern of convergence. The same is true when $x_0 = -0.8$.

When $x_0 = 0$, the sequence converges to the fixed point 0. In fact, since 0 is a fixed point, each term of the sequence is equal to 0, which is why no iterations can be seen on the diagram.

When $x_0 = 1.1$, the sequence tends to infinity. Only the first few iterations appear on the diagram, but you can see from the table that the terms of the sequence quickly become very large.

(b) When $x_0 = 0.999$, ten iterations are insufficient to indicate clearly the long-term behaviour of x_n. When the number of iterations is increased to 100, the diagram shows the sequence approaching the fixed point 0. The table provides further evidence for this behaviour, since it shows that the terms x_{90} to x_{100} all have value 0 to three decimal places.

(c) When $x_0 = 1.001$ and $N = 100$, Mathcad marks $f(x_n)$ (above the graph) in red, and neither the table for x_n nor the iteration diagram appears. Clicking on $f(x_n)$ reveals an error message which shows that some of the terms of the sequence are too large for Mathcad to cope with. This suggests that the sequence is unbounded. To obtain a clear idea of the behaviour of the sequence, it is helpful to reduce the number of iterations, as suggested in part (d).

(d) The terms in the table quickly become very large, which suggests that x_n tends to infinity; this behaviour is also suggested by the diagram.

Mathcad notes

Remember that Mathcad notes are *optional*.

- ◇ The error that arose in part (c) occurs when the result of a calculation exceeds in magnitude the largest positive number that Mathcad can handle, which is about 10^{307}.
- ◇ The expressions entered on the graph axes to plot the graphical iteration diagram have been 'hidden'. This has been done so that the graph can fit on the page! The expressions on the axes of any graph can be hidden by clicking in the graph to select it, then choosing **Graph ▶ X-Y Plot...** from the **Format** menu, selecting the 'Traces' tab and clicking in the check box to 'Hide Arguments'.

Double-clicking in the graph gives an alternative approach.

- ◇ The 'staircases' and 'cobwebs' are drawn on the graph by plotting x_{k+1} against x_k using the trace type 'step' (see Figure 4.1).

  Mathcad plots the sequence of points (x_k, x_{k+1}), for $k = 0, 1, 2, \ldots$, and joins each point (x_k, x_{k+1}) to (x_{k+1}, x_{k+2}) using a horizontal line segment ending at (x_{k+1}, x_{k+1}) followed by a vertical line segment. A separate trace is required to draw the first (vertical) line in the diagram from $(x_0, 0)$ to (x_0, x_1). This is done by plotting $x_1 \cdot (k > 0)$ against x_0 using the default trace type 'lines'. Mathcad assigns the inequality $(k > 0)$ in this expression the value 0 when it is false (for example, when $k = 0$) and 1 when it is true (for example, when $k = 1$).

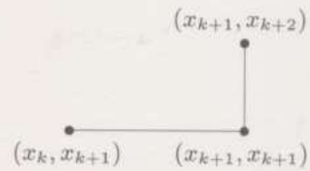


Figure 4.1 Trace type 'step'

In the next activity you will begin to investigate how the sequences obtained by iterating the function $f(x) = x^2 + c$ change as c is varied. So that the investigation is conducted in a systematic manner, you should keep the initial term x_0 set to the same value from now on. A suitable value for x_0 is 0, and this is the value that will be used in the remainder of the investigation. Thus you will be investigating how the behaviour of the iteration sequence

$$x_0 = 0, \quad x_{n+1} = x_n^2 + c \quad (n = 0, 1, 2, \ldots) \tag{4.1}$$

varies as c is varied. Throughout Subsections 4.1 and 4.2, the notation x_n will always mean a sequence of the form defined in equation (4.1).

For some values of c, the long-term behaviour of the sequence x_n may not be immediately obvious. As you work through the activity, you will be introduced to some ways in which you can adjust the values of variables on the Mathcad page to help you determine the long-term behaviour.

Activity 4.2 Exploring iteration sequences

In each of parts (a) to (e) you are asked to determine the long-term behaviour of the sequence x_n for a particular value of c, and then for two further values of c close to the first value. All of these values are marked on the number line in Figure 4.2, overleaf.

You should still be working with Mathcad file 221B1-01.

An 'F' has been marked above the number 0 on the number line, to indicate that when $c = 0$ the sequence tends to a fixed point. As you work through the activity, record your findings for the other values of c in a similar manner. Use a '2' if the sequence seems to tend to a 2-cycle, a '3' for a 3-cycle, and so on. If the sequence seems to tend to infinity, then use the symbol '∞'. If it seems to be chaotic, that is, if it does not appear to settle down to any steady long-term behaviour, then use the letter 'X'.

You looked at the case $c = 0$ in Activity 4.1(a).

First ensure that $x_0 = 0$. It is convenient to set the number of iterations and graph scale initially as follows: $N = 10$, $s1 = -2$ and $s2 = 2$.

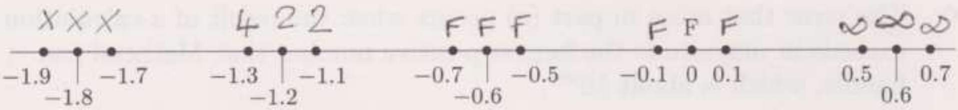


Figure 4.2 Values of c on the number line

(a) Set $c = 0.6$.

Record (on Figure 4.2) the long-term behaviour of the iteration sequence. Then repeat the process with $c = 0.7$ and $c = 0.5$.

(b) Set $c = 0.1$.

You may find it helpful to rescale the diagram, so that you can see the behaviour near the fixed point in more detail; try setting $s1 = -0.2$ and $s2 = 0.2$.

Record the long-term behaviour of the sequence. Then repeat the process with $c = -0.1$.

(c) Reset $s1 = -2$ and $s2 = 2$. Set $c = -0.6$.

You may find it helpful to increase the number of iterations and rescale the diagram; try $N = 100$, $s1 = -0.8$ and $s2 = 0.2$.

Record the long-term behaviour of the sequence. Then repeat the process with $c = -0.5$ and $c = -0.7$. When $c = -0.7$, you may find it helpful to increase N still further; try $N = 200$.

(d) Reset $s1 = -2$, $s2 = 2$ and $N = 10$. Set $c = -1.2$.

The long-term behaviour is not clear from the first ten iterations. However, the fixed-point and 2-cycle information below the graph shows that f has two repelling fixed points and an attracting 2-cycle, so the sequence may tend to this 2-cycle. To check this, increase the number of iterations; set $N = 100$.

The values in the table do indeed appear to be settling down. To see the long-term behaviour clearly on the diagram, you can arrange for only the later iterations to be plotted. The graph is set up to omit the first M iterations from the plot, so you just need to increase the value of M; try $M = 50$.

Record the long-term behaviour of the sequence. Then repeat the process with $c = -1.1$ and $c = -1.3$.

(e) Reset $N = 10$ and $M = 0$. Set $c = -1.8$.

Again, the long-term behaviour of the sequence is not clear. The fixed-point and 2-cycle information shows that although f has two fixed points and a 2-cycle, these are all repelling.

Increase the value of N to see if the behaviour of the sequence seems to settle down. Try $N = 100$, to begin with, and then try increasing N still further, say to 200, and then to 300. You should find that new construction lines appear each time you increase N, so it seems that the sequence does not settle down. It appears to be chaotic.

Notice, however, that this sequence appears to be bounded: all terms seem to lie between about -1.8 and 1.5.

Record the long-term behaviour of the sequence. Then repeat the process with $c = -1.7$ and $c = -1.9$.

Solutions are given on page 34.

In Activity 4.2 you were introduced to some ways in which you can adjust the values of variables in the Mathcad worksheet to help you find the long-term behaviour of an iteration sequence. These methods will be useful in the next activity. It is not possible to state an infallible strategy, but the list of suggestions in the box below should be a useful reference.

Identifying the long-term behaviour of an iteration sequence

The graphical iteration diagram, the table of terms, and the fixed-point and 2-cycle information can all help you to identify the long-term behaviour of an iteration sequence. Try the following suggestions. At any stage it may be helpful to rescale the diagram by adjusting the values of $s1$ and $s2$.

To explore whether the sequence tends to a fixed point

- ◇ Check that f has a fixed point, and that it is attracting (check that $|f'(a)| < 1$, where a is the fixed point).
- ◇ Increase the number N of iterations until the sequence settles down and appears to tend to the fixed point.

The sequence has 'settled down' when increases to the number of iterations do not produce any visible new construction lines (assuming that no large area of the diagram is already completely covered by construction lines!).

To explore whether the sequence tends to a 2-cycle

- ◇ Check that f has a 2-cycle, and that it is attracting (check that $|f'(a)f'(b)| < 1$, where a and b are the members of the 2-cycle).
- ◇ Increase the number of iterations until the sequence settles down. Then increase the number M of early iterations omitted from the plot until the remaining construction lines form a clear square.

To explore whether the sequence tends to a p-cycle ($p > 2$)

Increase the number of iterations until the sequence settles down. Then increase the number of early iterations omitted from the plot until only the more settled iterations are plotted. If possible, find the number of members of the cycle by counting the vertical construction lines in the diagram.

If some members of the cycle are close together, then the graphical iteration diagram may not be clear enough to allow you to count them. Rescaling parts of the diagram may help.

To explore whether the sequence is unbounded

Increase the number of iterations until the table shows that the terms become very large and positive or very large and negative, or the calculation for $f(x_n)$ gives the error 'Found a number with a magnitude greater than 10^307 while trying to evaluate this expression.'.

To explore whether an iteration sequence is chaotic

Increase the number of iterations (to, say, 100, then 200, then 300) and check that the sequence does not seem to settle down (check that new construction lines appear on the diagram each time the number of iterations is increased).

An apparently chaotic sequence may in fact tend to a cycle of very large order, but we ignore this possibility.

From the results obtained in Activity 4.2, it appears that between the numbers -2 and 1 there may be a range of values of c for which the sequence x_n tends to a fixed point, and a range of values of c for which it tends to a 2-cycle. In the next activity you are asked to find these ranges more precisely.

Activity 4.3 Attracting fixed points and attracting 2-cycles

You should still be working with Mathcad file 221B1-01.

By testing further values of c in the Mathcad worksheet, try to determine the ranges of values of c between -2 and 1 for which the sequence x_n behaves as follows:

(a) x_n tends to a fixed point;

(b) x_n tends to a 2-cycle.

It may help to record the results of these tests with your earlier results in Figure 4.2.

Solutions are given on page 34.

In Activity 4.3 you saw a range of values of c for which the sequence x_n tends to a fixed point, and a range of values of c for which it tends to a 2-cycle. We can also state a range of values of c for which the sequence x_n tends to infinity. As c increases, the graph of $y = x^2 + c$ moves vertically upwards. For $c > 0.25$, all parts of the graph lie above the line $y = x$, and graphical iteration shows that the iteration sequence with initial term $x_0 = 0$ tends to infinity. This is illustrated by the graphical iteration diagram for $c = 0.5$ in Figure 4.3. Thus, for all c in the open interval $(0.25, \infty)$, the sequence x_n tends to infinity.

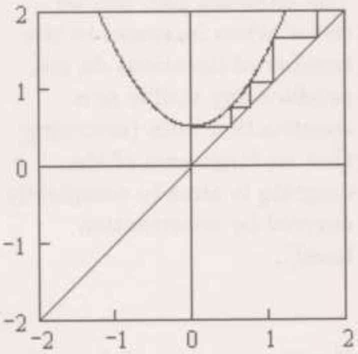


Figure 4.3 The case $c = 0.5$

In Activity 4.2, you saw that for $c = -1.3$ the sequence x_n tends to a 4-cycle, and you may have come across other values of c that give 4-cycles as you worked on Activity 4.3(b). If you wished, you could now go on to try to determine a range of values of c for which the sequence x_n tends to a 4-cycle, and then perhaps go on to look at other values of c. However, for c less than about -1.4, the way in which the behaviour of the sequence varies as c varies is very complicated. Chaotic behaviour is possible, but attracting cycles can also occur, as you will see in the next activity.

Activity 4.4 Attracting p-cycles

You should still be working with Mathcad file 221B1-01.

Set $s1 = -2$, $s2 = 2$, $N = 100$ and $M = 50$.

(a) Set c to each of the following values in turn, and in each case state the long-term behaviour of the iteration sequence x_n:

$$-1.39, -1.475, -1.575, -1.76.$$

(b) Set $c = -1.4$, and try to decide whether the iteration sequence is chaotic or tends to a p-cycle.

Solutions are given on page 34.

Comment

Once you have found a p-cycle that appears to be attracting, it is possible to check this classification of the p-cycle by calculating the corresponding gradient product; see the main text.

The final, *optional*, activity in this subsection uses the Mathcad worksheet to illustrate the fact that if the function f has a p-cycle for some value of p, then the members of the p-cycle are all fixed points of the composite function f^p.

Recall that f^p denotes the pth iterate of f, that is, the composite function $f \circ f \circ \cdots \circ f$, obtained when f is applied p times; see the main text.

When the variable p in the Mathcad worksheet is set to an integer greater than 1 (and not too large), the graph of f^p is added to the graphical iteration diagram.

Activity 4.5 p-cycles and the function f^p (Optional)

Set $s1 = -2$, $s2 = 2$, $N = 100$ and $M = 50$.

You should still be working with Mathcad file 221B1-01.

(a) Set $c = -1.2$ and $p = 2$. Observe from the graph of f^2 that the members of the 2-cycle are attracting fixed points of f^2.

(b) Repeat the process for the following values of c and p:

$$c = -1.3,\ p = 4; \quad c = -1.475,\ p = 6; \quad c = -1.76,\ p = 3.$$

You have already considered these values of c, in Activities 4.2 and 4.4.

Mathcad notes

The function f^p is displayed on the graph only when $p = 2, 3, 4, \ldots$. Its suppression when p is 0 or 1 is achieved by multiplying the expressions entered on the graph axes to plot the graph of f^p by $(p > 1)$. Mathcad assigns the expression $(p > 1)$ the value 0 when it is false and 1 when it is true.

Now close Mathcad file 221B1-01.

You may have wondered how we obtained the values of c given in Activity 4.4. It is not easy to find such attracting cycles by testing 'random' values of c; if you experimented using the Mathcad worksheet with values of c less than about -1.4, then you probably found that you usually obtained a sequence whose behaviour appeared to be chaotic. The next subsection shows how to *find* values of c that give cycles.

4.2 The bifurcation diagram

The diagram in Figure 4.4, overleaf, provides an overview of the changes in the long-term behaviour of the sequence x_n as c is varied. The diagram was generated, using Mathcad, by considering 1501 values of c, equally-spaced on the horizontal axis. For each of these values of c, the iteration sequence

Recall that the sequence x_n is defined by the recurrence system in equation (4.1).

$$x_0, x_1, x_2, \ldots, x_{300}$$

was calculated. The first 200 terms of this sequence were then ignored, and the remaining terms displayed on the graph, by plotting each of the points

$$(c, x_{200}), (c, x_{201}), \ldots, (c, x_{300}),$$

as a small dot.

If you pick a value of c, and imagine a vertical line drawn through the point on the horizontal axis of the diagram corresponding to that value of c, then the points where that vertical line intersects the graph give an indication of the long-term behaviour of the sequence x_n.

A single intersection point occurs when the plotted terms of the sequence are very close together. Two intersection points occur when each plotted term is close to one of two values. Three or more intersection points occur in a similar way.

If there is a single point of intersection, then this suggests that the sequence tends to a fixed point. Two intersection points suggest that it tends to a 2-cycle, 3 points suggest a 3-cycle, and so on. Many points suggest chaotic behaviour. You can see from the diagram, for example, that values of c in the interval $(-1.25, -0.75)$ seem to yield sequences x_n that tend to a 2-cycle. This agrees with the solution to Activity 4.3(b).

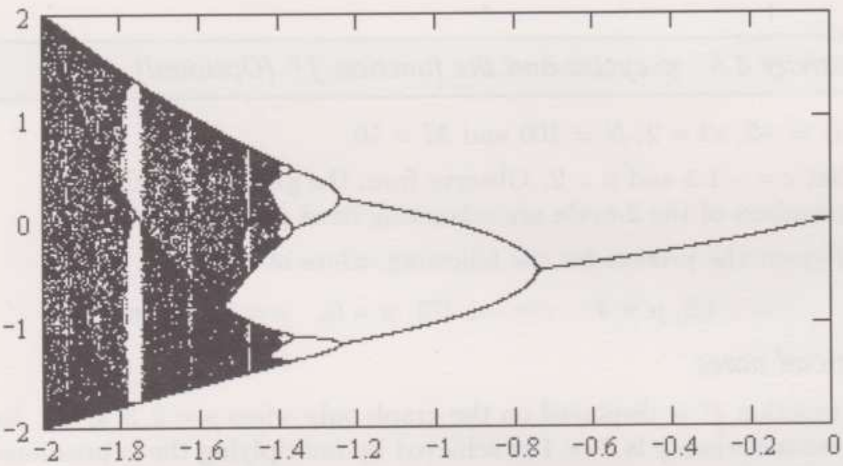


Figure 4.4 A bifurcation diagram for functions of the form $f(x) = x^2 + c$

The diagram illustrates that there are certain key values of c where the behaviour of the sequence x_n changes. For example, as c decreases through -0.75, the sequence stops tending to a fixed point, and begins to tend to a 2-cycle. Similarly, as c decreases through -1.25, the sequence stops tending to a 2-cycle, and begins to tend to a 4-cycle. At such values of c, each member of the cycle 'forks' into two cycle members, and we say that a *bifurcation* occurs. The graph in Figure 4.4 is known as a *bifurcation diagram.*

To 'bifurcate' is to fork into two.

At each bifurcation the number of members of the cycle doubles, since *every* member of the cycle 'forks' into two. For example, as c decreases from the value 0, sequences tending to fixed points give way to sequences tending to 2-cycles, which give way to sequences tending to 4-cycles, and so on. Similarly, as c decreases from the value -1.75, sequences tending to 3-cycles give way to sequences tending to 6-cycles, and so on. There are in fact infinitely many 'windows' in which the sequences tend to cycles and doubling of the number of members of the cycles occurs; in each of them the bifurcation points become progressively closer together as c decreases.

Outside these windows, sequences show bounded chaotic behaviour. For example, when $c = -1.8$, the sequence appears to be chaotic, with all terms lying between approximately -1.8 and 1.5; this confirms what you saw in Activity 4.2(e). You can see from the bifurcation diagram that as c decreases, the interval of values within which terms of the sequence lie gradually expands.

In the next activity you will use Mathcad to explore the bifurcation diagram. You will be asked to try to find a value of c that gives an iteration sequence which tends to a 5-cycle.

Activity 4.6 The bifurcation diagram

Open Mathcad file **221B1-02 Overview of iterations of f(x)=x^2+c**. This worksheet draws a bifurcation diagram for functions of the form $f(x) = x^2 + c$, with initial term $x_0 = 0$, for values of c from $C1$ to $C2$. You may have to wait for some time (perhaps 30 seconds) while Mathcad performs the calculations needed to draw the graph.

The bifurcation diagram is produced by taking $V + 1$ equally-spaced values of c. For each of these values of c, N iterations are carried out, and the first M of these are omitted from the plot. The worksheet is set up with $V = 500$, $N = 300$ and $M = 200$.

The bifurcation diagram in Figure 4.4 was produced by taking $V = 1500$, $N = 300$ and $M = 200$.

You can see parts of the diagram in more detail by altering the values of the horizontal axis limits $C1$ and $C2$. You can also increase the value of V; this produces a clearer diagram, but the calculations take longer.

If you seek to change both $C1$ and $C2$, then you will find that Mathcad starts to recalculate after the first change has been made. The Comment below gives ways to deal with this problem.

Try to find a value of c with an attracting 5-cycle, by proceeding as follows.

(a) First set $C2 = -1.5$, so that you can see the part of the diagram corresponding to values of c between -2 and -1.5 in more detail.

(b) Look at the part of the diagram between -1.65 and -1.6. You should see a narrow window that appears to include values of c for which the sequence x_n tends to a 5-cycle. Set $C1 = -1.65$ and $C2 = -1.6$ to confirm this observation, and choose a value of c that gives a 5-cycle.

(c) If you wish, test your value of c in Mathcad file 221B1-01.

A solution to part (b) is given on page 34.

Comment

◇ You can interrupt a Mathcad calculation by pressing [Esc], the escape key, and then clicking on **OK** in the resulting option box. Mathcad will recalculate when you next make a change.

◇ By default, Mathcad operates in 'automatic calculation mode', but this can be inconvenient where more than one input change is to be made before a time-consuming recalculation is required. In order to switch to manual calculation mode, which disables automatic calculation, select **Automatic Calculation** from the **Math** menu. (When you have done this, the tick mark beside **Automatic Calculation** in the menu disappears, and the word 'AUTO' disappears from the status bar in the bottom right corner of the Mathcad window.) Once in manual mode, you can calculate results when you choose, either by selecting **Calculate** from the **Math** menu, or by pressing the [F9] function key.

More information about this technique, and details of how to interrupt and resume calculations, are provided in *A Guide to Mathcad*.

In order to return to automatic mode, follow the same procedure, whereupon the tick mark reappears.

Mathcad notes

◇ The graph is produced by plotting $x_{i,n}$ against c_i using the trace type 'points'. The subscripted variable $x_{i,n}$ is entered in the usual way. Either use the 'x_n' button on the 'Matrix' toolbar or type [(left square bracket), then separate the subscripts i and n with a comma. Thus you could obtain $x_{i,n}$ on the screen by typing x[i,n.

$x_{i,n}$ is the notation used in the worksheet for the nth term of the sequence obtained using the ith value of c, denoted by c_i.

◇ A Mathcad graph can display at least 150 000 individual points. If you try to plot a graph with more points than this, then an error may occur. (No graph is drawn and the graph box is highlighted in red – clicking on it reveals the error message 'Can't plot this many points.'.)

Now close Mathcad file 221B1-02.

4.3 Other quadratic iteration sequences (Optional)

If you have studied Chapter B1 of MST121, then you may recognise that this sequence is associated with the *logistic recurrence relation*, and you will have seen the diagram in Figure 4.5 in connection with this.

Other families of quadratic functions also show complicated behaviour when they are iterated. Consider, for example, the family of iteration sequences

$$x_0 = 0.5, \quad x_{n+1} = x_n + rx_n(1 - x_n) \quad (n = 0, 1, 2, \ldots), \tag{4.2}$$

obtained by iterating the quadratic function $f(x) = x + rx(1 - x)$, where r is a parameter. Different values of r determine different iteration sequences from this family, in the same way that different values of c determine different iteration sequences from the family in equation (4.1). Figure 4.5 shows a bifurcation diagram for this new family, with r between 1 and 3. The structure of the graph appears essentially the same as that of the bifurcation diagram in Figure 4.4, although it is reversed and distorted.

Essentially the same bifurcation diagram occurs for various families of iteration sequences, generated both by quadratic functions and by other functions.

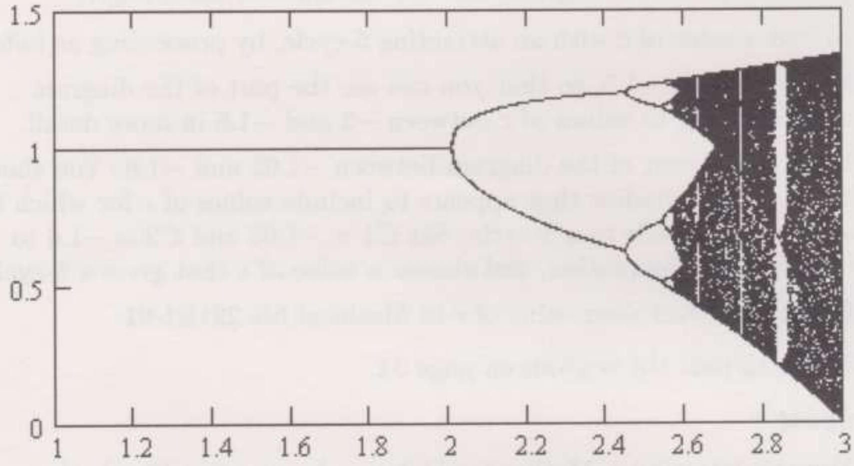


Figure 4.5 A bifurcation diagram for functions of the form $f(x) = x + rx(1 - x)$

The similarity of the bifurcation diagrams for these two families can be explained as follows. Suppose that the sequence x_n satisfies equation (4.2). Consider the new sequence

$$x'_n = -rx_n + \tfrac{1}{2}(1 + r) \quad (n = 0, 1, 2, \ldots).$$

This sequence x'_n is related to x_n by scaling (by the factor $-r$) and shifting (by adding $\frac{1}{2}(1 + r)$). Thus both the sequences x_n and x'_n have the same *type* of long-term behaviour (for example, if x_n tends to a 3-cycle, then x'_n tends to a 3-cycle). Now, it can be shown that the new sequence x'_n satisfies

You can check this, if you wish, by expressing x_n in terms of x'_n and substituting for x_0, x_n and x_{n+1} in equation (4.2).

$$x'_0 = 0.5, \quad x'_{n+1} = (x'_n)^2 + c \quad (n = 0, 1, 2, \ldots), \tag{4.3}$$

where $c = \frac{1}{4}(1 - r^2)$. Equation (4.3) is of the same form as equation (4.1), except that the initial term has changed from 0 to 0.5. It can be shown that this change of initial term does not affect the long-term behaviour of these sequences. Thus the long-term behaviour seen on a vertical slice through the graph in Figure 4.5, for some particular value of r, is of the same *type* as that seen on the vertical slice through the graph in Figure 4.4 for the corresponding value $c = \frac{1}{4}(1 - r^2)$.

Now as the parameter r takes values increasing from 1 to 3, the parameter $c = \frac{1}{4}(1 - r^2)$ takes values decreasing from 0 to -2. Thus the patterns in Figure 4.5 occur in the reverse order to those in Figure 4.4. They are also more 'squashed together' at one end because $\frac{1}{4}(1 - r^2)$ decreases progressively faster as r increases from 1 to 3. Other distortions occur because the scaling and shifting factors relating x_n to x'_n vary as r varies.

4.4 Sequences obtained by iterating real functions

In this subsection you will use Mathcad file 221B1-03, which allows you to explore the sequences obtained by iterating *any* real function.

A strategy for using this file is given in the box below.

If we wish to study the long-term behaviour of the sequences obtained by iterating a given real function f, then it is useful to find and classify any fixed points and 2-cycles of f. The Mathcad worksheet uses the Mathcad *solve block* to find approximate solutions to the fixed point equation and the 2-cycle equation. The solve block can be used whether or not these equations have solutions given by formulas. You have to provide Mathcad with a 'guess' for a solution; it then uses an iterative method to calculate a sequence of values that become progressively closer to a solution. Mathcad usually obtains a solution accurate to several decimal places; if it cannot find one, then it registers an error. Different initial guesses may yield different solutions.

If you have done the computer work for Chapter A3 of MST121, then you will have met the *solve block*. It is also covered in *A Guide to Mathcad*.

To classify the fixed points and 2-cycles of a function f, we have to be able to find the gradient of the graph of f at the fixed points, and at members of the 2-cycles. The Mathcad worksheet uses a built-in feature of Mathcad, the $\mathrm{d}/\mathrm{d}x$ operator, to find an approximate value for the gradient, here denoted by the expression $\mathrm{D}f(x)$. You will learn much more about this topic in Block C; for now, just accept the values that Mathcad provides!

In Block C, you will see that the gradient $f'(x)$ of the graph of a function f at $(x, f(x))$ is also represented by the notation

$$\frac{\mathrm{d}}{\mathrm{d}x}(f(x)),$$

called the *derivative* of f at x.

In the next activity you are asked to explore iterations of a particular real function, using the following strategy.

This strategy should be read while you are doing Activity 4.7 overleaf.

Exploring iteration sequences using Mathcad file 221B1-03

Edit the definition of the function f, on page 2, as required.

To find and classify all the fixed points of f (page 2)

1. If necessary, rescale the graph to show all the fixed points of f. (Adjust the values of the axis limits $s1$ and $s2$.)
2. Use the graph to estimate the approximate value of a fixed point of f, and set the 'guess' value x to this value. Read off the more accurate value a given for the fixed point. Use the value of the gradient $f'(a)$ to classify the fixed point.
3. Repeat the instructions in step 2 as many times as is necessary to find and classify all the fixed points of f.

The value a is indicated on the graph by a small black box at the point (a, a). If a is not the value that you wished to find, then try to set the 'guess' value x more accurately.

To find and classify all the 2-cycles of f (page 3)

1. If necessary, rescale the graph to show all the fixed points of the composite function $f \circ f$. Identify the fixed points of $f \circ f$ that are not fixed points of f (these are the points where the graph of $f \circ f$ meets the line $y = x$ but the graph of f does not). These points pair off into 2-cycles of f.
2. Use the graph to estimate the approximate value of a fixed point of $f \circ f$ that is not a fixed point of f, and set the 'guess' value x to this value. Read off the more accurate value a, and the corresponding value $b = f(a)$, for the 2-cycle a, b of f. Use the value of the gradient product $f'(a)f'(b)$ to classify the 2-cycle.
3. Repeat the instructions in step 2 as many times as is necessary to find and classify all the 2-cycles of f.

The values a and b are indicated on the graph by small black boxes at the points (a, a) and (b, b), joined by dashed black lines. If a is not the value that you wished to find, then try to set the 'guess' value x more accurately.

To identify the long-term behaviour of a sequence (page 4)

Use the suggestions given in the box on page 9 to help you to find the long-term behaviour of the iteration sequence.

The variables N, M, $s1$, $s2$ and p have the same roles here as in Subsection 4.1.

Activity 4.7 Iterating real functions

Open Mathcad file **221B1-03 Iterating real functions**. The function

$$f(x) = \frac{4x}{(1+x^2)^3}$$

is already entered for you on page 2 of the worksheet. Use the worksheet and the strategy on page 15 to help you to do the following.

(a) Find all the fixed points of f, and classify them as attracting, repelling or indifferent.

(b) Find all the 2-cycles of f, and classify them as attracting, repelling or indifferent.

(c) For each of the following initial terms x_0, find the long-term behaviour of the sequence obtained by iterating f with that initial term:

$$x_0 = -1, \quad x_0 = 1.$$

Solutions are given on page 34.

Mathcad notes

◇ Both the solve block (used to find a solution of an equation) and the d/dx operator (used to calculate a gradient) use numerical methods to obtain approximations to the exact solution and gradient, respectively. The error messages 'No solution was found. ...' (for the solve block) and 'Can't converge to a solution.' (for the d/dx operator) indicate that the numerical method has failed. In the case of the solve block it is worth trying different initial guesses, though there may in fact be no solution. In the case of the d/dx operator, you could try to find the gradient at a nearby point.

◇ The accuracy of the solve block is controlled by the built-in variable TOL (**Math** menu, **Options...**, 'Built-In Variables' tab, Convergence Tolerance (TOL)). Mathcad looks for a solution until the difference between two successive approximations is less than or equal to TOL. By default, TOL = 0.001, but for this file it is set to 0.000 001. For the d/dx operator, the value given is usually accurate to 7 or 8 significant figures, irrespective of the value of TOL.

◇ The choice of the name Df for the gradient function of f in the Mathcad worksheet is a pragmatic one. The usual notation, f' ('f prime' or 'f dash'), could have been used, but the prime symbol is hard to see when placed to the right of an 'f' in Mathcad.

The prime symbol is obtained in Mathcad by pressing the left quote key.

Now close Mathcad file 221B1-03.

Chapter B2, Section 5
Visualising affine transformations

Mathcad provides a convenient means by which you can see the effects of linear and affine transformations on the unit square and unit grid.

5.1 Linear transformations

In the first activity you will explore linear transformations of the plane. You will use a Mathcad worksheet that allows you to enter any 2×2 matrix $\mathbf{A}$ and see the effect of the linear transformation $f(\mathbf{x}) = \mathbf{Ax}$ on the unit square and unit grid.

You can take f to be one of the basic linear transformations by setting $\mathbf{A}$ equal to one of the following matrices, expressed in 'Mathcad notation'.

This notation is used in all the Mathcad worksheets for Chapters B2 and B3.

Mathcad matrix notation for basic linear transformations

Linear transformation	**Mathcad matrix notation**
Rotation about the origin through angle θ	$R(\theta) = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$
Reflection in the line through the origin that makes angle θ with the positive x-axis	$Q(\theta) = \begin{pmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{pmatrix}$
Scaling with factor a in the x-direction and factor b in the y-direction	$S(a, b) = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$
Uniform scaling with factor a	$U(a) = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$
x-shear with factor a	$X(a) = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$
y-shear with factor a	$Y(a) = \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}$

For example, the matrix of a rotation about the origin through angle $\pi/2$ anticlockwise can be specified in the worksheet by entering '$R(\frac{\pi}{2})$'. Note that our Mathcad notation for rotation and reflection matrices differs slightly from the notation used in the main text. In the Mathcad notation, parameters appear in brackets (for example, $R(\frac{\pi}{2})$), whereas in the main text they appear as subscripts (for example, $\mathbf{R}_{\pi/2}$). The Mathcad notation is used not only in the Mathcad worksheets but also in this computer book.

In the first worksheet, the side of the unit square that lies along the x-axis is marked with a filled-in triangle, as shown in Figure 5.1(a), overleaf. The image of this triangle is plotted on the image square, so that you can tell which vertices of the square have been mapped to which vertices of the

image. This lets you see whether orientation has been preserved or reversed. For example, Figure 5.1(b) shows the image of the unit square under a rotation through $\pi/2$ anticlockwise about the origin, and Figure 5.1(c) shows its image under reflection in the y-axis.

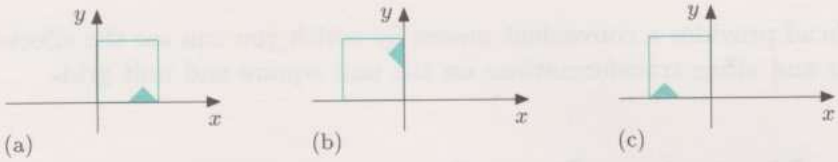


Figure 5.1 The unit square and two images

See Chapter B2, Section 2.

You saw in the main text that you can predict whether a linear transformation f preserves or reverses orientation by considering the determinant of its matrix $\mathbf{A}$. The transformation f preserves orientation if $\det \mathbf{A} > 0$ and reverses orientation if $\det \mathbf{A} < 0$. If $\det \mathbf{A} = 0$, then f is a flattening, which destroys orientation. The determinant of $\mathbf{A}$ also tells you the effect of f on areas: these are scaled by the factor $|\det \mathbf{A}|$.

Activity 5.1 Images of the unit square and unit grid

These matrix techniques are used in the computer work for Chapter B2 of MST121.

Open Mathcad file **221B2-01 Linear transformations**. Page 2 of the worksheet describes the basic techniques for creating, editing and calculating with matrices in Mathcad. Page 3 sets up and explains the notation used in the worksheet.

The determinant of $\mathbf{A}$ is denoted by $|\mathbf{A}|$ in Mathcad.

Move to page 4. Here a 2×2 matrix $\mathbf{A}$ is defined, and its determinant calculated. Graphs display the unit square and unit grid, and their images under the linear transformation $f(\mathbf{x}) = \mathbf{A}\mathbf{x}$. You can set $\mathbf{A}$ to be a matrix representing one of the basic linear transformations, using the notation explained before this activity. Alternatively, you can set $\mathbf{A}$ to be the general 2×2 matrix $\mathbf{M}$. The page is set up with $\mathbf{A} = R\left(\frac{\pi}{2}\right)$.

The edges of the graph box may be obscured in places by the unit grid.

The graphs can be rescaled by changing the value of s, which is defined near the top of the page; both axes of both graphs are scaled from $-s$ to s.

Set $\mathbf{A}$ to be each of the following matrices in turn. In each case, use the value of the determinant of $\mathbf{A}$ to predict whether the transformation preserves, reverses or destroys orientation, and whether it decreases, preserves or increases area, and try to confirm your predictions by looking carefully at the effect of the transformation on the unit square.

(a) Rotations: $R(\frac{\pi}{4})$, $R(-\frac{\pi}{4})$.

(b) Reflections: $Q(\frac{\pi}{4})$, $Q(\frac{3\pi}{4})$.

(c) Scalings: $S(2,1)$, $S(0.5,1)$, $S(-1,1)$, $S(-1,-3)$.

(d) Uniform scalings: $U(2)$, $U(0.5)$, $U(-1)$, $U(-0.5)$.

(e) x-shears: $X(2)$, $X(1)$, $X(0)$, $X(-1)$, $X(-2)$.

(f) y-shears: $Y(2)$, $Y(-1)$.

For parts (g) and (h), define $\mathbf{A} := \mathbf{M}$. The matrix in part (g) is already entered for you; edit $\mathbf{M}$ to obtain the matrices in part (h).

(g) A flattening: $\begin{pmatrix} 4 & -6 \\ -2 & 3 \end{pmatrix}$.

(h) Non-basic linear transformations: $\begin{pmatrix} 4 & -1 \\ -2 & 3 \end{pmatrix}$, $\begin{pmatrix} 0.4 & -1 \\ -0.4 & -0.4 \end{pmatrix}$.

Comment

- Parts (a) to (g) illustrate the effects of basic linear transformations on orientation and area, which can be summarised as follows.

  Orientation-preserving transformations: rotations, shears, scalings with factors a, b of the same sign.

  Remember that a uniform scaling $U(a)$ is a scaling, with both factors equal to a.

  Orientation-reversing transformations: reflections, scalings with factors a, b of opposite signs.

  Area-preserving transformations: rotations, reflections, shears, scalings with factors a, b where $|ab| = 1$.

  Area-decreasing transformations: scalings with factors a, b where $|ab| < 1$, flattenings.

  Area-increasing transformations: scalings with factors a, b where $|ab| > 1$.

  Flattenings destroy orientation and decrease area to zero.
- The linear transformation represented by the first matrix in part (h) preserves orientation and increases area; the second reverses orientation and decreases area.

We now consider general linear transformations. For example, Figure 5.2 shows the effect on the unit square and unit grid of the linear transformation m represented by the matrix $\mathbf{M} = \begin{pmatrix} 0 & -2 \\ 2 & 2 \end{pmatrix}$.

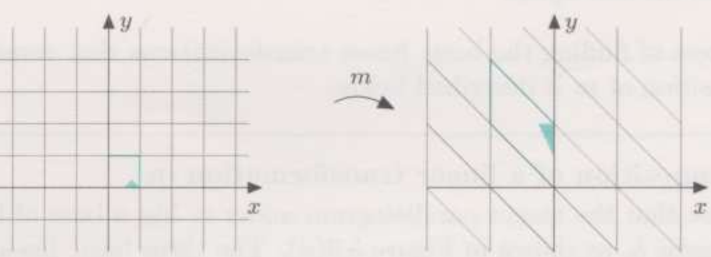


Figure 5.2 Image under the linear transformation m

This linear transformation is in fact the composite of a uniform scaling with factor 2, followed by an x-shear with factor 1, followed by a rotation through $\pi/2$, as we now check using matrix multiplication.

Recall that if f and g are linear transformations represented by matrices $\mathbf{A}$ and $\mathbf{B}$ respectively, then the composite linear transformation $g \circ f$ is represented by the product matrix $\mathbf{BA}$. It follows that if f, g and h are linear transformations of the plane represented by matrices $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ respectively, then the composite linear transformation $h \circ g \circ f$ is represented by the matrix $\mathbf{CBA}$.

See Chapter B2, Section 3. Remember that $g \circ f$ means 'first apply f, then apply g'.

If we take $\mathbf{A}$ to be the matrix representing a uniform scaling with factor 2, $\mathbf{B}$ to be the matrix representing an x-shear with factor 1, and $\mathbf{C}$ to be the matrix representing a rotation through $\pi/2$, then

$$\mathbf{CBA} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 0 & -2 \\ 2 & 2 \end{pmatrix} = \mathbf{M},$$

as required.

Any linear transformation m of the plane can be expressed as a composite of at most three basic linear transformations. This shows that every linear transformation of the plane has a fairly simple geometrical interpretation. In particular, if m is not a flattening, then it can be expressed as the composite of a scaling, followed by an x-shear, followed by a rotation.

Some of the basic linear transformations in this 'decomposition' of m may be the identity transformation, in which case m is a composite of two basic linear transformations, or is itself a basic linear transformation.

The uniform scaling with factor 1, the x-shear with factor 0 and the rotation through angle 0 are all equal to the identity transformation.

To express a given linear transformation that is not a flattening as a composite of a scaling, followed by an x-shear, followed by a rotation, we consider its effect on the unit square. Figure 5.3(a) shows the unit square and Figure 5.3(d) shows its image under a typical linear transformation m. The image is a parallelogram with one vertex at the origin, which we call the *target parallelogram*. We use the word *base* to describe the side of the unit square that lies along the x-axis and the image of this side on the target parallelogram. These base sides are marked with filled-in triangles. Figure 5.3(b) and (c) illustrate stages in finding the decomposition, with the corresponding base sides marked similarly.

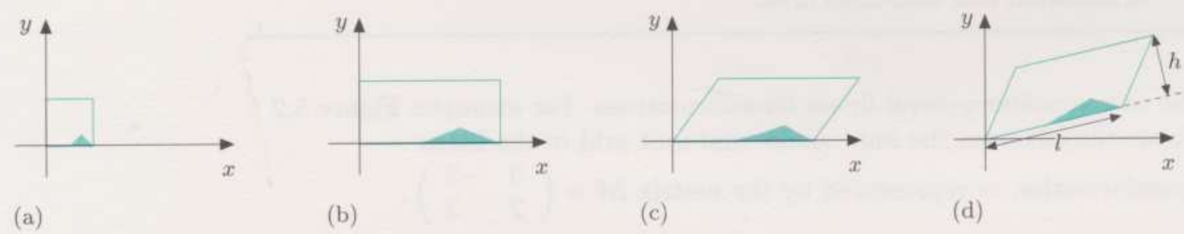


Figure 5.3 Successive images of the unit square

The process of finding the basic linear transformations that constitute the decomposition of m is described below.

Decomposition of a linear transformation m

Suppose that the target parallelogram under m has a base of length l, and height h, as shown in Figure 5.3(d). The three basic linear transformations for the decomposition can be chosen as follows.

1. First choose the scaling with matrix $S(\pm l, h)$, where the $+$ sign is used if m preserves orientation and the $-$ sign otherwise. This maps the unit square to a rectangle with base of length l and height h; see Figure 5.3(b).
2. Next take the x-shear that maps the rectangle onto a parallogram that can be rotated onto the target parallelogram; see Figure 5.3(c).
3. Finally, take the rotation about the origin that maps the parallelogram onto the target parallelogram in Figure 5.3(d).

This rectangle has the same orientation and area as the target parallelogram.

In step 2, you can find the x-shear required by determining the signed distance, d say, that the x-shear moves a point on the side of the rectangle opposite the base. The shear factor is then d/h.

Here d is positive if the point moves to the right and negative if it moves to the left.

In the next activity you will use a page of the Mathcad worksheet that is designed to help you carry out this decomposition process for a given linear transformation. Parts (a) and (b) of the activity provide examples, and in part (c) you are asked to work through the process yourself.

Activity 5.2 Composite linear transformations

Move to page 5 of the worksheet. Near the top of the page a matrix $\mathbf{M}$ is defined, and the value of its determinant $|\mathbf{M}|$ is displayed. Graphs display the unit square and unit grid, and their images under the linear transformation $m(\mathbf{x}) = \mathbf{Mx}$. The bases of the unit square and the target parallelogram (shown in blue) are marked with filled-in triangles.

You should still be working with Mathcad file 221B2-01.

Beneath this, the page is set up so that you can define the matrix $\mathbf{A}$ of a scaling, the matrix $\mathbf{B}$ of an x-shear, and the matrix $\mathbf{C}$ of a rotation. The determinants of $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{CBA}$ are all displayed. Initially, we have

$$\mathbf{A} = S(1,1), \quad \mathbf{B} = X(0), \quad \mathbf{C} = R(0). \tag{5.1}$$

Note that each of the matrices $S(1,1)$, $X(0)$ and $R(0)$ represents the identity transformation.

The linear transformations represented by $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ are defined as f, g and h, respectively. The images of the unit square and unit grid under f, $g \circ f$, and $h \circ g \circ f$ are displayed in successive graphs. Each of these graphs also shows the target parallelogram, in dashed blue lines. All the graphs can be rescaled by changing the value of s, defined near the top of the page.

(a) The page is set up with

$$\mathbf{M} = \begin{pmatrix} 0 & -2 \\ 2 & 2 \end{pmatrix}.$$

The decomposition of this matrix $\mathbf{M}$ was discussed earlier in the subsection.

In this case $|\mathbf{M}|$ is positive, so m preserves orientation.

Set each of the matrices $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ as shown below, in turn, and as you do so check that they achieve steps 1, 2 and 3, respectively, of the decomposition process on page 20.

$$\mathbf{A} = S(2,2), \quad \mathbf{B} = X(1), \quad \mathbf{C} = R(\tfrac{\pi}{2}).$$

(b) Now reset each of the matrices $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ as in equations (5.1). Then edit the entries of the matrix $\mathbf{M}$ to give

$$\mathbf{M} = \begin{pmatrix} -1 & -1 \\ 0 & 2 \end{pmatrix}.$$

In this case $|\mathbf{M}|$ is negative, so m reverses orientation.

Set each of the matrices $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ as shown below, in turn, and as you do so check that they achieve steps 1, 2 and 3, respectively, of the decomposition process on page 20.

$$\mathbf{A} = S(-1,2), \quad \mathbf{B} = X(-\tfrac{1}{2}), \quad \mathbf{C} = R(0).$$

(c) In each of parts (i) to (iv) below, first reset each of the matrices $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ as in equations (5.1), and set $\mathbf{M}$ to the given matrix. Then follow the decomposition process to express m as a composite of basic linear transformations. Hence express each given matrix as a product of matrices of basic linear transformations.

For each of these matrices, the parameters of the required scaling and shear are small integers or simple fractions, and the parameter of the required rotation is an integer multiple of $\pi/2$.

(i) $\begin{pmatrix} 2 & 2 \\ 0 & 1 \end{pmatrix}$ (ii) $\begin{pmatrix} -2 & -2 \\ 0 & -1 \end{pmatrix}$ (iii) $\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$ (iv) $\begin{pmatrix} 0 & 3 \\ 3 & 1 \end{pmatrix}$

Solutions to part (c) are given on page 35.

Now close Mathcad file 221B2-01.

5.2 Affine transformations

In the next activity you will use Mathcad to visualise the effect of some affine transformations of the plane. You will be given the images of the unit square and unit grid under various 'mystery' affine transformations, and your task is to find the affine transformations that produce these images.

Activity 5.3 Exploring affine transformations

Open Mathcad file **221B2-02 Affine transformations**. Page 2 of the worksheet defines the matrices representing the basic linear transformations, using the Mathcad notation.

Move to page 3. The unit square and unit grid, and their images under a 'mystery' affine transformation, are displayed near the top of the page. The page can be set up with any one of four different mystery transformations, and you can choose which one of these is applied by setting the variable T to 1, 2, 3 or 4. All of the graphs can be rescaled by changing the value of s, defined near the top of the page.

You can define your own affine transformation $f(\mathbf{x}) = \mathbf{A}\mathbf{x} + \mathbf{a}$ by editing the definitions of the matrix $\mathbf{A}$ and vector $\mathbf{a}$ near the middle of the page. The unit square and unit grid, and their images under f, are displayed immediately below these definitions. The graph that displays the images under f also shows the image of the unit square under the mystery affine transformation, in dashed blue lines; this is the target.

In each of the given examples, $\mathbf{A}$ is the matrix of a basic linear transformation, and the vector $\mathbf{a}$ has integer components.

Set T to each of 1, 2, 3 and 4 in turn. In each case, try to identify the mystery affine transformation, and verify your answer by setting $\mathbf{A}$ and $\mathbf{a}$ and checking that you hit the target.

Solutions are given on page 35.

Comment

You may wish to use the Mathcad page to see the effects of other affine transformations on the unit square and unit grid. If you set $T = 0$, then no target will appear on the graphs.

Mathcad notes

The definitions of the four mystery affine transformations are hidden in the columns of a matrix off to the right of page 3 of the worksheet. The area beyond the right-hand margin of a Mathcad page (which is marked by a solid vertical line) can be used just like the rest of the worksheet. It is divided into further pages, where you can place mathematical expressions, text, graphs and pictures. Some of the other Mathcad worksheets in the course also have material off the page to the right.

However, it is not necessary to view this area in order to carry out the activities based on the worksheet, nor to understand the principles behind them.

You can view the pages in this area by using the horizontal scroll bar to move to the right. When printing a wide worksheet, you can choose whether or not the content to the right of the left-hand pages is to be printed. To do this, select **Page Setup...** from the **File** menu. Then, in the 'Page Setup' option box, tick or untick the check box for 'Print single page width', as required. If there is a tick here, then just the left-hand pages of the worksheet will be printed when **Print...** is selected subsequently; this is the default for MS221 worksheets. Otherwise, the whole width of the worksheet will be printed.

Given any three points, in any order, there is a unique affine transformation f that maps the points $(0,0)$, $(1,0)$, $(0,1)$ to these points, respectively. The main text gave a method for finding f. The next, *optional*, activity allows you to explore the effect of affine transformations on the triangle with vertices $(0,0)$, $(1,0)$, $(0,1)$, which we call the *unit triangle*. This lets you check visually, using Mathcad, that an affine transformation found using the given method does indeed map the three points $(0,0)$, $(1,0)$, $(0,1)$ to the required image points.

See Chapter B2, Section 4.

Activity 5.4 Applying affine transformations to the unit triangle (Optional)

Move to page 4 of the worksheet. Here you can define an affine transformation $f(\mathbf{x}) = \mathbf{A}\mathbf{x} + \mathbf{a}$ by setting the matrix $\mathbf{A}$ and vector $\mathbf{a}$ as described in the worksheet. Graphs display the unit grid and unit triangle, and their images under f. The coordinates of the vertices of the image triangle, and its area, are shown at the bottom of the page.

You should still be working with Mathcad file 221B2-02.

Use the Mathcad page to verify the correctness of the affine transformations found in Example 4.1 and Activity 4.2 in the main text.

Mathcad notes

The '$|x|$' button (available on both the 'Calculator' and 'Matrix' toolbars), whose keyboard alternative is `[Shift]\` (shift and backslash), performs several roles in Mathcad. It gives the determinant of a matrix, the modulus (absolute value) of a real number, the magnitude of a vector, and the modulus of a complex number.

You will study complex numbers later in the course.

The modulus of the determinant of a matrix $\mathbf{A}$ (written as $|\det \mathbf{A}|$ in the main text) can be obtained by two applications of '$|x|$', and appears as $||\mathbf{A}||$.

Now close Mathcad file 221B2-02.

Chapter B3, Section 5
Iterating linear transformations with the computer

In this section you will use Mathcad to explore sequences of points in the plane obtained by iterating linear transformations. You will also see a few examples of sequences obtained by iterating affine transformations. The section ends with an *optional* subsection in which you can see some visually interesting plots obtained by an iteration process involving more than one affine transformation.

This notation is given on page 17.

The Mathcad notation for matrices of basic linear transformations used in the computer work for Chapter B2 will be used again in this section, both in the text and in the Mathcad files.

5.1 Iterating linear transformations

This subsection is concerned with sequences of points in the plane obtained by iterating linear transformations. In the main text, you saw some examples of such sequences where the matrix representing the linear transformation has two distinct non-zero eigenvalues, the case of a so-called *generalised scaling*. These have the following properties.

See Chapter B3, Section 4.

Iteration properties of generalised scalings

Let the linear transformation f be represented by a 2×2 matrix $\mathbf{A}$ that has two distinct non-zero eigenvalues k_1 and k_2, with corresponding eigenlines ℓ_1 and ℓ_2. Let (x_0, y_0) be a point of $\mathbb{R}^2$ and let (x_n, y_n) be an iteration sequence generated by $\mathbf{A}$, with initial point (x_0, y_0).

(a) (i) If $k_1 > 0$, then (x_n, y_n) all lie on the same side of ℓ_2 as (x_0, y_0).

(ii) If $k_1 < 0$, then (x_n, y_n) alternate between opposite sides of ℓ_2.

(b) (i) If $\max\{|k_1|, |k_2|\} > 1$, then the sequence moves away from $(0, 0)$.

(ii) If $\max\{|k_1|, |k_2|\} < 1$, then the sequence moves towards $(0, 0)$.

(c) If $|k_1| > |k_2|$ and (x_0, y_0) does not lie on an eigenline, then

$$\frac{y_n}{x_n} \to m \text{ as } n \to \infty,$$

where m is the gradient of ℓ_1.

Property (c) states that if k_1 is the 'dominant eigenvalue', then (x_n, y_n) tends in the direction of the 'dominant eigenline' $y = mx$.

The first activity involves sequences obtained by iterating non-uniform scalings. These are the 'simplest' generalised scalings. The scaling with factors a and b, where $a \neq b$, has eigenvalues a and b and the corresponding eigenlines are $y = 0$ (the x-axis) and $x = 0$ (the y-axis), respectively.

For example, Figure 5.1 shows the first few points of the iteration sequence generated by $\mathbf{A} = S(1.5, 1)$ and the initial point $(2, 1)$. It illustrates behaviour that can be predicted using the iteration properties of generalised scalings given in the box above. Since both eigenvalues are positive, all points of the sequence lie on the same side of the eigenline $y = 0$, and on the same side of the eigenline $x = 0$, as the initial point. Since $\max\{|1.5|, |1|\} > 1$, the sequence moves away from $(0, 0)$.

Since $|1.5| > |1|$, the dominant eigenvalue is 1.5, the dominant eigenline is $y = 0$, and the sequence tends in the direction of this line.

It follows from property (b) of generalised scalings (see Chapter B3, Section 3) that this sequence remains the same distance from the line $y = 0$ and moves away from the line $x = 0$.

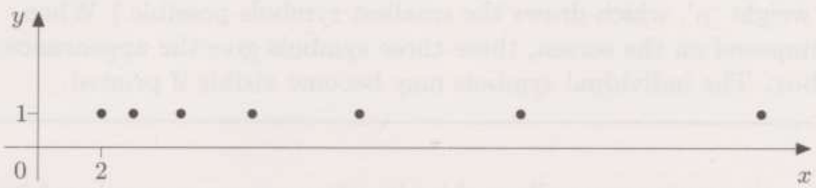


Figure 5.1 Iteration sequence generated by a non-uniform scaling

In Activity 5.1 you will see sequences obtained by iterating various non-uniform scalings.

Activity 5.1 Non-uniform scalings

Open Mathcad file **221B3-01 Iterating linear transformations.** Page 2 of the worksheet defines the matrices representing basic linear transformations, using the Mathcad notation.

Move to page 3. Here a 2×2 matrix $\mathbf{A}$ and an initial point (x_0, y_0) are defined; the initial point is defined as the vector $\mathbf{x}_0 = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$.

In this computer book,
$$\mathbf{x}_n = \begin{pmatrix} x_n \\ y_n \end{pmatrix}$$
for $n = 0, 1, 2, \ldots$.

A graph displays the first point, as a black box symbol, and N subsequent points, as magenta box symbols, of the iteration sequence $\mathbf{x}_{n+1} = f(\mathbf{x}_n)$, where $f(\mathbf{x}) = \mathbf{Ax}$. A solid magenta box distinguishes $\mathbf{x}_N$, the final point calculated. Mathcad also calculates and plots the eigenlines of the matrix $\mathbf{A}$ (if there are any); if there are two eigenlines and one is dominant, then the dominant one is shown in red, and the other in blue. You can rescale the graph by altering the value of s.

You can set $\mathbf{A}$ to be a matrix representing one of the basic linear transformations (or a product of such matrices). Alternatively, you can set $\mathbf{A}$ to be the 2×2 matrix $\mathbf{M}$, whose entries you can edit.

In this activity, you will only set $\mathbf{A}$ to be a scaling.

The page is set up with $\mathbf{A} = S(1.5, 1)$, $(x_0, y_0) = (2, 1)$, $N = 1$ and $s = 25$.

Figure 5.1 illustrates the sequence generated by this choice of $\mathbf{A}$ and (x_0, y_0).

(a) Set N to 2, 3 and 4 in turn, so the third, fourth and fifth points of the sequence appear on the graph. (Making the points appear one at a time shows the order in which they appear in the sequence.)

(b) Set N to 10, 20 and 30 in turn, and describe the effect on the value of the ratio y_N/x_N, which is displayed to the left of the graph. Explain how you could have predicted this.

When you do this, only a few points of the sequence appear on the graph, because the subsequent points lie outside the graph box. You can rescale the graph to see more points, if you wish.

(c) Set the initial point (x_0, y_0) to $(-2, 1)$. Observe that the iteration sequence now tends in the negative direction of the x-axis.

Experiment with a few other initial points of your own. In each case, predict the behaviour of the sequence and check your prediction.

(d) Reset the initial point to $(2, 1)$, and ensure that $N = 30$ and $s = 25$.

Set $\mathbf{A}$ to each of the non-uniform scaling matrices below in turn, and in each case check that the behaviour of the sequence is as predicted by the iteration properties of generalised scalings.

$$S(1.5, 1.2),\ S(1.5, 0.8),\ S(1.5, -1.2),\ S(1.5, -0.8),\ S(0.8, -0.8).$$

For the last of these matrices, rescale the graph by setting $s = 3$.

Solutions to parts (b) and (c) are given on page 35.

Mathcad notes

The solid box that marks the point (x_N, y_N) on the graph is obtained by plotting *three* traces, using a box, a '×' and a '+'. (The traces are plotted using weight 'p', which draws the smallest symbols possible.) When superimposed on the screen, these three symbols give the appearance of a solid box. The individual symbols may become visible if printed.

In the next activity you will consider iteration sequences produced by generalised scalings for which the eigenlines are different from the axes. You have already seen some examples of such sequences in the main text. For example, you considered the linear transformation with matrix

See Chapter B3, Section 3.

$$\begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix},$$

which has eigenvalues 4 and -1, with corresponding eigenlines $y = \frac{3}{2}x$ and $y = -x$, respectively. By the iteration properties of generalised scalings, the long-term behaviour of iteration sequences whose initial point is not on the eigenlines can be described in this case as follows.

The points of such an iteration sequence alternate between opposite sides of the eigenline $y = \frac{3}{2}x$ and lie on the same side of the eigenline $y = -x$ as the initial point. The sequence moves away from $(0, 0)$ and tends in the direction of the dominant eigenline $y = \frac{3}{2}x$.

Figure 5.2 shows the first few points of the iteration sequence with initial point $(2, 1)$.

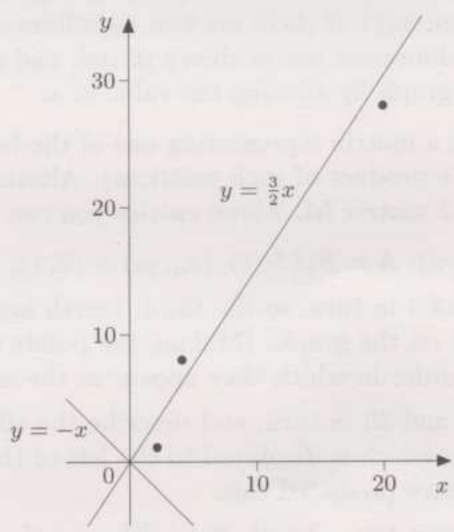


Figure 5.2 Iteration sequence generated by a generalised scaling

The next activity involves examples of a similar kind.

Activity 5.2 Linear transformations with two distinct eigenvalues

Ensure that the number of iterations, initial point and graph scale on page 3 of the worksheet are set as follows:

You should still be working with Mathcad file 221B3-01.

$$N = 30, \quad \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad s = 100.$$

In each of parts (a) to (d) below, a matrix $\mathbf{A}$ is given, together with its eigenvalues and eigenlines. In each case, use this information together with the iteration properties of generalised scalings to predict the behaviour of the iteration sequence $\mathbf{x}_n = \mathbf{A}^n\mathbf{x}_0$ with initial point $(2, 1)$, giving a description along the lines of the example discussed before this activity. Then confirm your answer by setting $\mathbf{A}$ appropriately in the Mathcad page and observing the graph.

In part (a), define $\mathbf{A} := \mathbf{M}$; the matrix $\mathbf{M}$ is already entered for you. For the matrices in parts (b) to (d), edit the entries of $\mathbf{M}$.

For some of these matrices, you may find it helpful to set N to 1, 2, 3 and so on, to see the order in which the points appear.

(a) $\begin{pmatrix} -2 & 1 \\ 4 & 1 \end{pmatrix}$
has eigenvalues 2 and -3, with eigenlines $y = 4x$ and $y = -x$, respectively.

(b) $\begin{pmatrix} 1 & 3 \\ -1 & 5 \end{pmatrix}$
has eigenvalues 4 and 2, with eigenlines $y = x$ and $y = \frac{1}{3}x$, respectively.

(c) $\begin{pmatrix} 0.9 & 0.1 \\ -0.6 & 0.4 \end{pmatrix}$
has eigenvalues 0.7 and 0.6, with eigenlines $y = -2x$ and $y = -3x$, respectively.

For parts (c) and (d), set $s = 3$.

(d) $\begin{pmatrix} -0.6 & 0.5 \\ 0.9 & 0.6 \end{pmatrix}$
has eigenvalues 0.9 and -0.9, with eigenlines $y = 3x$ and $y = -\frac{3}{5}x$, respectively.

Solutions are given on page 35.

In the next activity you will see some examples of sequences obtained by iterating linear transformations whose matrices have no eigenvalues. (Recall that when we say in this course that a matrix has no eigenvalues, we mean that it has no *real* eigenvalues. It will have *complex* eigenvalues, but we are not concerned with those.) Such sequences show a wide variety of behaviour, and you are not expected to explore them in detail.

Activity 5.3 Linear transformations with no eigenvalues

You should still be working with Mathcad file 221B3-01.

Ensure that the number of iterations, initial point and graph scale on page 3 of the worksheet are set as follows:

$$N = 200, \quad \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad s = 5.$$

The matrices listed in parts (a) to (d) below have no eigenvalues; nor therefore do they have any eigenvectors. For each matrix, use your geometric knowledge of basic linear transformations to try to predict the general 'shape' of the iteration sequence produced by the matrix. (This is not easy for the matrices in parts (c) and (d)!) Then confirm your answer by setting $\mathbf{A}$ appropriately in the Mathcad page.

For some of these matrices, you may find it helpful to set N to 1, 2, 3 and so on, to see the order in which the points appear.

(a) Rotations:

$$R(\tfrac{\pi}{6}), \quad R(\tfrac{\pi}{3}), \quad R(1).$$

When setting $\mathbf{A}$ to a product of two matrices, remember to type $*$ (or use the '×' button on the 'Calculator' toolbar) to indicate the multiplication.

(b) Rotations composed with uniform scalings:

$$U(0.9)\,R(\tfrac{\pi}{12}), \quad U(-0.9)\,R(\tfrac{\pi}{12}), \quad U(1.01)\,R(\tfrac{\pi}{12}).$$

(c) A rotation composed with a non-uniform scaling:

$$S(0.8, 1.2)\,R(\tfrac{\pi}{12}).$$

(d) Rotations composed with shears:

$$X(0.2)\,R(\tfrac{\pi}{12}), \quad X(-0.6)\,R(\tfrac{\pi}{12}).$$

Solutions are given on page 36.

Comment

No matrix of the types in parts (a) and (b), representing rotations, or composites of rotations and uniform scalings, has any eigenvalues (except where the angle of rotation is an integer multiple of π). However, some matrices of the types in parts (c) and (d), representing composites of rotations and non-uniform scalings, or composites of rotations and shears, can have two eigenvalues. For example, $S(0.8, 1.4)\,R(\frac{\pi}{12})$ and $X(0.3)\,R(\frac{\pi}{12})$ both have two eigenvalues.

In the final, *optional*, activity in this subsection, you can see a few examples of sequences obtained by iterating linear transformations whose matrices have just one eigenvalue. Again, such sequences show a wide variety of behaviour, but it is worth looking at one particular aspect of this behaviour. You have seen that if $\mathbf{A}$ is a matrix with *two* eigenvalues of different magnitudes, and the initial point $\mathbf{x}_0$ does not lie on an eigenline of $\mathbf{A}$, then

$$\frac{y_n}{x_n} \to m \text{ as } n \to \infty,$$

where (x_n, y_n) is the $(n+1)$th point of the sequence $\mathbf{x}_n = \mathbf{A}^n\mathbf{x}_0$ and m is the gradient of the dominant eigenline. In the next activity you are invited to explore whether a similar property holds for 2×2 matrices with just one eigenvalue.

Activity 5.4 Linear transformations with one eigenvalue (Optional)

Ensure that the initial point and graph scale on page 3 of the worksheet are set as follows:

You should still be working with Mathcad file 221B3-01.

$$\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad s = 50.$$

Each of the six matrices in parts (a) and (b) below has just one eigenvalue and just one eigenline. In each case set $N = 50$, set $\mathbf{A}$ to the given matrix, and note the gradient of the eigenline. Then increase N to 500, and check whether the value of the ratio y_N/x_N seems to be approaching the gradient of the eigenline.

The gradient of the eigenline is the y-component of the eigenvector with x-component equal to 1.

For some of these matrices, you will need to rescale the graph to obtain a clear picture of the iteration sequence.

(a) (i) $X(4)$ (ii) $U(1.5)\,X(4)$ (iii) $U(0.5)\,X(4)$

(b) (i) $\begin{pmatrix} -3 & 1 \\ -4 & 1 \end{pmatrix}$ (ii) $\begin{pmatrix} 0.5 & -1 \\ 1 & 2.5 \end{pmatrix}$ (iii) $\begin{pmatrix} -1.4 & 0.2 \\ -0.2 & -1 \end{pmatrix}$

In part (b), define $\mathbf{A} := \mathbf{M}$ and edit the entries of $\mathbf{M}$.

Solutions are given on page 36.

Comment

Most 2×2 matrices with just one eigenvalue have just one eigenline. The only exceptions are the matrices of uniform scalings and the zero matrix; for these matrices, *every* line through the origin is an eigenline.

It can be shown that every linear transformation of the plane whose matrix has only one eigenvalue is a composite of a shear and a uniform scaling. If the matrix has only one eigenline, then the shear is parallel to this eigenline.

One or both of the transformations forming the composite may be the identity transformation.

Mathcad notes

In part (a)(iii), you may have noticed that for both $N = 50$ and $N = 500$ Mathcad displays the last point calculated as $(x_N, y_N) = (0, 0)$ but gives a non-zero value for the ratio y_N/x_N, despite the fact that Mathcad (somewhat dubiously) evaluates $0/0$ as 0. The reason for this is that, while Mathcad always retains values to 15 significant figures for calculation purposes, in this worksheet numbers of magnitude less than 10^{-10} are displayed as zero on the screen. So Mathcad displays the values of $x_{50} = 1.794 \times 10^{-13}$ and $y_{50} = 8.882 \times 10^{-16}$ as zero, while displaying the calculated value of y_{50}/x_{50} as 4.95×10^{-3}.

If you wish to display such small values, then click in an empty space in the worksheet to obtain the red cross cursor, and select **Result...** from the **Format** menu. In the 'Result Format' option box, choose the 'Tolerance' tab, and increase the value of 'Zero threshold'. For example, increasing this value from 10 to 100 will ensure that only numbers of magnitude less than 10^{-100} are displayed as zero.

Now close Mathcad file 221B3-01.

5.2 Iterating affine transformations

In Subsection 5.1 we looked at sequences of points in the plane obtained by iterating linear transformations. In the next activity you will see a few examples of iteration sequences obtained by iterating affine transformations. You saw in Chapter B2 that an affine transformation of the plane is a function of the form

$$\begin{aligned} f\colon \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ \mathbf{x} &\longmapsto \mathbf{A}\mathbf{x} + \mathbf{a}, \end{aligned}$$

where $\mathbf{A}$ is a 2×2 matrix and $\mathbf{a}$ is a vector with two components.

We generate an iteration sequence using an affine transformation f in a similar way to the other iteration sequences that you have seen: we choose an initial point $\mathbf{x}_0$ and repeatedly use the recurrence relation

$$\mathbf{x}_{n+1} = f(\mathbf{x}_n) \quad (n = 0, 1, 2, \ldots).$$

Activity 5.5 Affine transformations

Open Mathcad file **221B3-02 Iterating affine transformations**. Page 2 of the worksheet defines the matrices representing basic linear transformations, using the Mathcad notation.

Move to page 3. This is similar to page 3 of the worksheet for Mathcad file 221B3-01, but it is set up to allow you to define a vector $\mathbf{a}$ as well as a matrix $\mathbf{A}$, and the function iterated is $f(\mathbf{x}) = \mathbf{A}\mathbf{x} + \mathbf{a}$. The initial point (x_0, y_0) is set to $(2, 1)$, and the number N of iterations is set to 10.

(a) The matrix $\mathbf{A}$ is set to the rotation matrix $R(\frac{\pi}{6})$, and the vector $\mathbf{a}$ is set to $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$, so f is initially a linear transformation.

Set $N = 20$ and describe the effect on the iteration sequence.

Then set $\mathbf{a}$ to each of the following vectors in turn and describe the effect on the iteration sequence:

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \end{pmatrix}.$$

If you are not sure about the order in which the points appear in a sequence, then set N to 1, 2, 3 and so on, in turn, to find out.

(b) Set $\mathbf{A} = U(0.9)\,R(\frac{\pi}{6})$, which represents a rotation composed with a uniform scaling, and set $\mathbf{a} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

Set $N = 40$ and describe the effect on the iteration sequence.

Then set $\mathbf{a}$ to each of the vectors in part (a) and describe the effect on the iteration sequence.

Solutions are given on page 36.

Now close Mathcad file 221B3-02.

It can be shown that if the 2×2 matrix $\mathbf{A}$ represents an anticlockwise rotation through an angle θ about the origin, then each affine transformation $f(\mathbf{x}) = \mathbf{Ax} + \mathbf{a}$ is an anticlockwise rotation through the angle θ about some point whose coordinates depend on θ and $\mathbf{a}$. Thus if $\mathbf{A}$ represents a rotation, then each affine transformation $f(\mathbf{x}) = \mathbf{Ax} + \mathbf{a}$ produces iteration sequences whose points lie on circles.

You saw in Chapter B2, Section 4, how to find $\mathbf{a}$ so that f is a rotation about a given point.

Similarly, if $\mathbf{A}$ represents a composite of a rotation and a uniform scaling, then each affine transformation $f(\mathbf{x}) = \mathbf{Ax} + \mathbf{a}$ produces iteration sequences that spiral about a point whose coordinates depend on θ and $\mathbf{a}$.

5.3 Iterated function systems (Optional)

In Subsection 5.2, you looked at some sequences of points in the plane obtained by iterating affine transformations. In this subsection, you will see some sequences obtained in a similar way, but using more than one affine transformation. A collection of transformations used to generate an iteration sequence in this way is known as an *iterated function system*. Such systems can give surprising results!

Do not be tempted to spend too long on this subsection!

See Activity 5.7, for example.

For example, consider the two affine transformations

$$f(\mathbf{x}) = \mathbf{Ax} + \mathbf{a} \quad \text{and} \quad g(\mathbf{x}) = \mathbf{Ax} - \mathbf{a},$$

where $\mathbf{A} = U(0.6)\,R(0.5)$ and $\mathbf{a} = \begin{pmatrix} 0.5 \\ 0 \end{pmatrix}$.

The matrix $\mathbf{A}$ represents a composite of a rotation and a uniform scaling. Both f and g produce spiral sequences if iterated on their own, the spirals centred on points other than the origin, since $\mathbf{a}$ is not the position vector of the origin. The sequences spiral inwards towards their centres, because the scaling factor has magnitude less than 1.

The angle of rotation is 0.5 radians (about 29°) and the scaling factor is 0.6.

We can produce an iteration sequence using *both* f and g by starting with an initial point, and choosing in some way between f and g at each iteration. For example, we could simply choose either f or g at random. Figure 5.3 shows a plot of the first 50 000 points of a sequence obtained by iterating f and g in this way; the initial point is the origin.

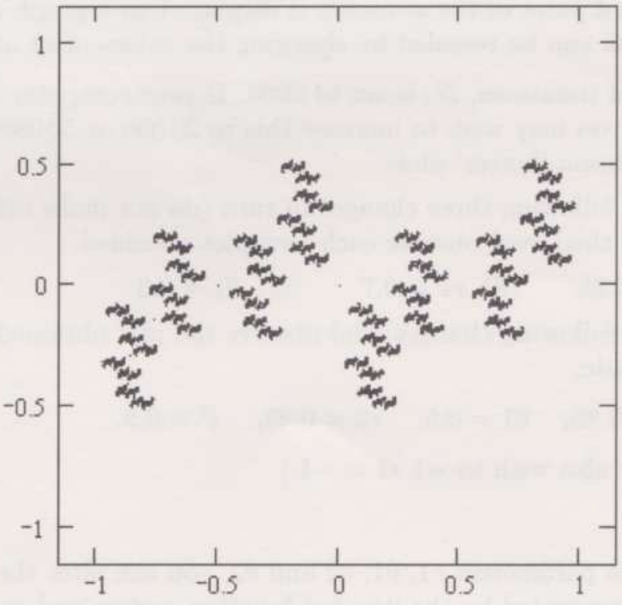


Figure 5.3 Sequence generated by an iterated function system

Complicated self-similar sets are often called *fractals.*

As you can see, the sequence produces a plot of a set that is of great complexity yet displays the property of *self-similarity*; that is, small parts of the set are 'similar' to large parts. Moreover, the set has an interesting 'natural' appearance, perhaps resembling some kind of sea life!

In the next activity you are invited to use a Mathcad page to see a variety of plots obtained in a similar way. The page allows you to define two affine transformations f and g, and an initial point $\mathbf{x}_0$. It generates and plots a sequence $\mathbf{x}_n$ using the following method. At each iteration, Mathcad generates a random number p_n between 0 and 1, and the next point of the sequence is calculated using the recurrence relation

$$\mathbf{x}_{n+1} = \begin{cases} f(\mathbf{x}_n), & \text{if } p_n < P, \\ g(\mathbf{x}_n), & \text{if } p_n \geq P, \end{cases}$$

where P is a number defined in the worksheet. The variable P is initially set to 0.5, so each of f and g is chosen approximately equally often.

Activity 5.6 Two-function systems (Optional)

Open Mathcad file **221B3-03 Iterated function systems**. Page 2 of the worksheet defines the matrices representing basic linear transformations, using the Mathcad notation.

Move to page 3. Here two affine transformations

$$f(\mathbf{x}) = \mathbf{Ax} + \mathbf{a} \quad \text{and} \quad g(\mathbf{x}) = \mathbf{Bx} + \mathbf{b},$$

are defined, where

$$\mathbf{A} = U(r1)\,R(\theta 1) \quad \text{and} \quad \mathbf{B} = U(r2)\,R(\theta 2).$$

The variables $r1$, $\theta 1$, $r2$ and $\theta 2$, and the vectors $\mathbf{a}$ and $\mathbf{b}$, are initially set to values that give the transformations f and g used to generate the sequence in Figure 5.3.

The initial point $\mathbf{x}_0$ is set to the origin. The iteration sequence $\mathbf{x}_n$ of points is generated by using a sequence p_n of random numbers and a variable P to decide which of the transformations f and g is used at each iteration, in the way described before this activity. The number of iterations carried out is N.

Each calculated point of the sequence is displayed on a graph as a small dot. The graph can be rescaled by changing the values of $s1$ and $s2$.

The number of iterations, N, is set to 5000. If your computer is fast enough, then you may wish to increase this to 20 000 or 50 000, for example, to obtain 'better' plots.

(a) Make the following three changes in turn (do not make other changes as you do this), and observe each new plot obtained.

(i) $r1 = 0.85$ (ii) $r1 = 0.7$ (iii) $\theta 1 = 0.3$

In part (b), Mathcad starts a new plot after each change. For ways to avoid waiting, see the Comment on page 13.

(b) Make the following changes, and observe the plot obtained after they are *all* made:

$$r1 = 0.95, \quad \theta 1 = 0.5, \quad r2 = 0.25, \quad P = 0.9.$$

(You may also wish to set $s1 = -1$.)

Comment

By varying the parameters $r1$, $\theta 1$, $r2$ and $\theta 2$, you can alter the shape of the sequence generated by the iterated function system and so produce a wide variety of 'natural-looking objects'.

Mathcad notes

- ◇ The Mathcad function **if** chooses one of two values depending on a condition. The expression **if**(condition, a, b) gives the value a if the condition is true, and b if the condition is false. Several **if** statements can be combined together to make multiple choices.

See *A Guide to Mathcad*.

- ◇ The Mathcad function **rnd** generates uniformly-distributed random numbers. The expression **rnd**(x) returns a random number between 0 and x. The values produced by the **rnd** function come from a sequence of random numbers generated by Mathcad when it starts up. The same sequence is generated every time, unless you change the random number generator. To do this, select **Options...** from the **Math** menu, choose the 'Built-In Variables' tab and then set the 'Seed value for random numbers'. Each choice of positive integer gives a different sequence; the default value is 1.
- ◇ The graphs in the worksheet are plotted using the trace type 'points', with symbol 'none'. Mathcad plots each point as a small dot.

It is a remarkable fact that the shape of the set obtained by this method is associated only with f and g, and not with the way that we choose between them at each iteration. Briefly, if the functions f and g are both *contracting*, that is, they reduce the distance between pairs of points, then there is a bounded set F of points which has the property that if we choose any point in F, and apply *either* f or g to this point, then we obtain another point in F. If we choose a point outside F, and repeatedly apply either f or g, then the resulting sequence approaches the set F, and may or may not eventually lie within F.

One way to obtain a picture of the set F is to iterate f and g in the random manner described. This usually produces a 'chaotic' sequence whose points, at least after the first few, are either in the set or very close to the set. For some pairs of transformations f and g, such as the pair in Activity 5.6(b), we need to ensure that one of the transformations is applied more often than the other if we want the points of the sequence to be fairly evenly distributed across the set.

The self-similar nature of sets like those in Activity 5.6 led mathematicians to the remarkable discovery that it is possible to use iterated function systems to generate a *given* set by analysing the ways in which it is self-similar. This usually requires more than two affine transformations. A classic example is the fern, which you can see in the final activity.

Activity 5.7 The fern: a four-function system (Optional)

Move to page 4 of the worksheet, and observe the plot obtained from the given iterated function system of four affine transformations.

You should still be working with Mathcad file 221B3-03.

If your computer is fast enough, then you may wish to increase the number of iterations N to 20 000 or 50 000, for example, to obtain a better plot.

If you wish to experiment with the plot, then you could change the rotation matrix in the definition of the matrix $\mathbf{A}$; for example, try $R(0)$ and then $R(0.05)$. Other small changes to the parameters in the matrices will give ferns of slightly different shapes.

Now close Mathcad file 221B3-03.

Solutions to Activities

Chapter B1

Solution 4.2

(a) When c takes any of the values 0.5, 0.6 or 0.7, the sequence tends to infinity.

(b) When c takes either of the values -0.1 or 0.1, the sequence tends to a fixed point.

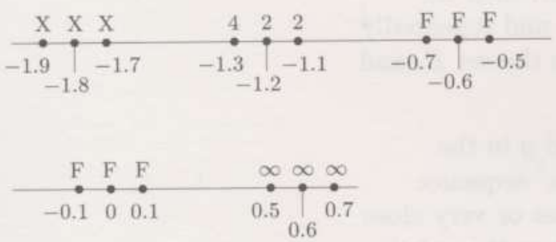
(c) When c takes any of the values -0.5, -0.6 or -0.7, the sequence tends to a fixed point.

(d) When c takes the value -1.3, the sequence tends to a 4-cycle.

When c takes either of the values -1.1 or -1.2, the sequence tends to a 2-cycle.

(e) When c takes any of the values -1.7, -1.8 or -1.9, the sequence seems to be chaotic.

The results of parts (a) to (e) are summarised in the following figure.



Solution 4.3

(a) The sequence x_n tends to a fixed point for all values of c in the interval $[-0.75, 0.25]$.

(For values of c inside this interval but close to its left end, this convergence is difficult to see from the graphical iteration diagram or the table, but it is confirmed by the fact that the gradient of the relevant fixed point lies between -1 and 1, so the fixed point is attracting. Note that if c is inside the interval but very close to either end, then the gradient at the attracting fixed point is displayed as ± 1, due to rounding.

When $c = 0.25$ and $c = -0.75$ the relevant fixed points are indifferent, and in both cases you have to set Mathcad to carry out a large number of iterations to see that the sequence appears to tend to the fixed point.

For values of c a little greater than 0.25, the sequence tends to infinity.

For values of c a little less than -0.75, both fixed points are repelling, and the sequence tends to a 2-cycle.)

(b) The sequence x_n tends to a 2-cycle for all values of c in the interval $[-1.25, -0.75)$.

(When $c = -1.25$ the relevant 2-cycle is indifferent, and you have to set Mathcad to carry out a large number of iterations to see that the sequence appears to tend to the 2-cycle. For values of c a little less than -1.25, the 2-cycle is repelling, and the sequence tends to a 4-cycle.)

Solution 4.4

(a) The behaviour of the sequence x_n for the given values of c is as follows:

-1.39: x_n tends to an 8-cycle;
-1.475: x_n tends to a 6-cycle;
-1.575: x_n tends to a 7-cycle;
-1.76: x_n tends to a 3-cycle.

(b) Setting $c = -1.4$ produces a graphical iteration diagram with a fairly large number of construction lines. Increasing the number of iterations does not produce any visible new construction lines, so it appears that the sequence tends to a cycle. The number of members of the cycle cannot be counted in the diagram scaled from -2 to 2, since some of the construction lines are too close together. (In fact, this is a 32-cycle.)

Solution 4.6

(b) Values of c between about -1.628 and -1.625 give 5-cycles.

Solution 4.7

(a) The three fixed points are 0, 0.766 and -0.766 (to three decimal places). They are all repelling.

(b) The two 2-cycles are -1.034, -0.467 and 1.034, 0.467 (to three decimal places). They are both attracting.

(c) When $x_0 = -1$, the sequence tends to the first 2-cycle given in the answer to part (b).

When $x_0 = 1$, the sequence tends to the second 2-cycle given in the answer to part (b).

Chapter B2

Solution 5.2

(c) (i) The matrix represents the composite of a scaling with factors 2 and 1, followed by an x-shear with factor 2; that is,

$$\begin{pmatrix} 2 & 2 \\ 0 & 1 \end{pmatrix} = X(2)\,S(2,1)$$
$$= \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}.$$

(You can check the answer by matrix multiplication.)

(ii) The matrix represents the composite of a scaling with factors 2 and 1, followed by an x-shear with factor 2, followed by a rotation through the angle π; that is,

$$\begin{pmatrix} -2 & -2 \\ 0 & -1 \end{pmatrix} = R(\pi)\,X(2)\,S(2,1)$$
$$= \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}.$$

(iii) The matrix represents the composite of an x-shear with factor -1, followed by a rotation through the angle $\pi/2$; that is,

$$\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} = R(\tfrac{\pi}{2})\,X(-1)$$
$$= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}.$$

(iv) The matrix represents the composite of a scaling with factors -3 and 3, followed by an x-shear with factor $-\frac{1}{3}$, followed by a rotation through the angle $\frac{3\pi}{2}$ (or, equivalently, $-\frac{\pi}{2}$); that is,

$$\begin{pmatrix} 0 & 3 \\ 3 & 1 \end{pmatrix} = R(\tfrac{3\pi}{2})\,X(-\tfrac{1}{3})\,S(-3,3)$$
$$= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & -\frac{1}{3} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -3 & 0 \\ 0 & 3 \end{pmatrix}.$$

Solution 5.3

Target figure 1 is obtained by setting

$$\mathbf{A} = R(\tfrac{\pi}{4}) \quad\text{and}\quad \mathbf{a} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}.$$

Target figure 2 is obtained by setting

$$\mathbf{A} = Q(0) \quad\text{and}\quad \mathbf{a} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}.$$

Target figure 3 is obtained by setting

$$\mathbf{A} = S(2,3) \quad\text{and}\quad \mathbf{a} = \begin{pmatrix} -3 \\ -4 \end{pmatrix}.$$

Target figure 4 is obtained by setting

$$\mathbf{A} = Y(2) \quad\text{and}\quad \mathbf{a} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}.$$

Chapter B3

Solution 5.1

(b) The ratio y_N/x_N appears to tend to 0 as N increases. This could have been predicted because 0 is the gradient of the dominant eigenline $y = 0$.

(c) If the initial point lies to the right of the y-axis, then the sequence tends in the positive direction of the x-axis, whereas if the initial point lies to the left of the y-axis, then the sequence tends in the negative direction of the x-axis.

(The dominant eigenvalue 1.5 is positive, so each point of the sequence lies on the same side of the non-dominant eigenline as the initial point.)

If the initial point lies on the y-axis, then subsequent points of the sequence coincide with the initial point. (In this case, Mathcad highlights the ratio y_N/x_N in red and displays no value for it, since its calculation would require division by zero.)

Solution 5.2

(a) The points of the sequence alternate between opposite sides of the eigenline $y = 4x$ and stay on the same side of the eigenline $y = -x$ as the initial point. The sequence moves away from $(0,0)$ and tends in the direction of the dominant eigenline $y = -x$.

(b) The points of the sequence stay on the same side of the eigenline $y = x$ and on the same side of the eigenline $y = \frac{1}{3}x$ as the initial point. The sequence moves away from $(0,0)$ and tends in the direction of the dominant eigenline $y = x$.

(c) The points of the sequence stay on the same side of the eigenline $y = -2x$ and on the same side of the eigenline $y = -3x$ as the initial point. The sequence moves towards $(0,0)$ and tends in the direction of the dominant eigenline $y = -2x$.

(d) The points of the sequence stay on the same side of the eigenline $y = -\frac{3}{5}x$ as the initial point, and alternate between opposite sides of the other eigenline $y = 3x$. The sequence moves towards $(0,0)$. Neither eigenline is dominant.

(In fact, the points lie alternately on each of a pair of lines through the origin.)

Solution 5.3

(a) The matrix $R(\frac{\pi}{6})$ gives a sequence each of whose points is one of twelve points equally spaced around a circle centred at the origin.

The matrix $R(\frac{\pi}{3})$ gives a sequence each of whose points is one of six points equally spaced around a circle centred at the origin.

The matrix $R(1)$ gives a sequence whose points lie around a circle centred at the origin; no points are repeated.

(b) The matrix $U(0.9)\,R(\frac{\pi}{12})$ gives a sequence whose points lie on a spiral centred at the origin; each point is closer to the origin than its predecessor.

The matrix $U(-0.9)\,R(\frac{\pi}{12})$ gives a sequence whose points lie alternately on each of two spirals centred at the origin; each point is closer to the origin than its predecessor.

The matrix $U(1.01)\,R(\frac{\pi}{12})$ gives a sequence whose points lie on a spiral centred at the origin; each point is further from the origin than its predecessor.

(c) The matrix $S(0.8, 1.2)\,R(\frac{\pi}{12})$ gives a sequence whose points lie on an 'elliptical spiral', centred at the origin and spiralling towards the origin.

(d) The matrix $X(0.2)\,R(\frac{\pi}{12})$ gives a sequence whose points lie on an ellipse centred at the origin.

The matrix $X(-0.6)\,R(\frac{\pi}{12})$ also gives a sequence whose points lie on an ellipse centred at the origin.

Solution 5.4

The value of the ratio y_N/x_N seems to approach the gradient of the eigenline in all six cases.

(It is indeed true that if $\mathbf{A}$ is a 2×2 matrix with just one eigenvalue and just one eigenline, and $\mathbf{x}_0$ is a point in the plane, then

$$\frac{y_n}{x_n} \to m \text{ as } n \to \infty,$$

where (x_n, y_n) is the $(n+1)$th point of the iteration sequence $\mathbf{x}_n = \mathbf{A}^n\mathbf{x}_0$ and m is the gradient of the eigenline.)

Solution 5.5

(a) Setting $N = 20$ illustrates that after the first twelve points the sequence begins to repeat. Each point in the sequence is one of twelve points equally spaced on a circle centred at the origin. (This sequence was the first of those considered in Activity 5.3(a).)

Changing $\mathbf{a}$ to a vector other than the position vector of the origin produces a sequence whose points lie on a circle centred at a point other than the origin.

(Changing $\mathbf{a}$ usually also changes the radius of the circle, since this is the distance between the centre of the circle and the initial point.)

(b) Setting $N = 40$ plots more points of the sequence, spiralling towards the origin.

Changing $\mathbf{a}$ to a vector other than the position vector of the origin produces a sequence that spirals inwards about a point other than the origin.

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